Haunted Kaluza Universe with Four-dimensional Lorentzian Flat, Kerr, and Taub–NUT Slices

Rossen I. Ivanov† and Emil M. Prodanov*

†*Department of Pure and Applied Mathematics, Trinity College, University of Dublin, Ireland, and Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko chaussee, Sofia – 1784, Bulgaria*
e-mail: ivanovr@tcd.ie

**School of Physical Sciences, Dublin City University, Glasnevin, Ireland*
e-mail: prodanov@physics.dcu.ie

Abstract

The duality between the original Kaluza's theory and Klein's subsequent modification is duality between slicing and threading decomposition of the five-dimensional spacetime. The field equations of the original Kaluza's theory lead to the interpretation of the four-dimensional Lorentzian Kerr and Taub–NUT solutions as resulting from static electric and magnetic charges and dipoles in the presence of ghost matter and constant dilaton, which models Newton's constant.

1 Introduction

What is referred today as Kaluza–Klein theory (see [1] for an extensive collection of papers) is Klein's modification [2] of the original Kaluza's theory [3]. These two theories are dual [4, 5] and the duality between them is the duality between threading and slicing decomposition [6] of the five-dimensional spacetime — foliation with one-dimensional or one-codimensional surfaces. The field equations [7] of Klein's theory (threading decomposition) express Newton's constant as a dynamical field (dilaton) and do not allow a constant solution for the dilaton unless an unphysical restriction to the Maxwell tensor F_{ij} is imposed: namely, $F_{ij}F^{ij} = 0$ (Latin indexes run from 1 to 4, Greek — from 1 to 5). In 1983 Gross *et al.* [8] and Sorkin [9] found magnetic monopoles in Klein's theory by considering four-dimensional Euclidean and periodic in time Kerr [10] and Taub–NUT [11, 12] solutions which were trivially embedded into a vacuum five-dimensional Klein's universe with timelike fifth dimension. The original Euclidean periodic time was then identified as the fifth dimension and the magnetic vector potentials — as former degrees of freedom of the four-dimensional Kerr or Taub–NUT solution. The resulting four-dimensional gravity has a non-constant dilaton and has lost the original Kerr or Taub–NUT geometry.
Unlike Klein's theory, in the original Kaluza's theory (slicing decomposition) the gauge degrees of freedom of the electromagnetic potentials A_i are transferred to the dilaton [4, 5]. This allows us to consider four-dimensional spacetimes with constant dilaton (i.e. Newton's constant) and fixed gauge. We show that a constant dilaton and a vacuum five-dimensional Kaluza universe necessitate a Ricci-flat four-dimensional slice. In the dual Kaluza's setup, the Kerr or Taub–NUT geometry of the four-dimensional slice is preserved. The field equations of the original Kaluza's theory lead to the interpretation of the four-dimensional Lorentzian Kerr and Taub–NUT solutions as resulting from static electric and magnetic charges and dipoles in the presence of ghost matter.

2 Field Equations of Kaluza's Theory

The five-dimensional Kaluza's metric is:

$$G_{\mu\nu} = \left(\begin{array}{c|c} g_{ij} & A_i \\ \hline A_i & \phi \end{array}\right) \tag{1}$$

The five-dimensional interval in mostly-plus metric is:

$$d\sigma^2 = g_{ij}(dy^i + A^i ds)(dy^j + A^j ds) + \frac{1}{N^2} ds^2 \,, \tag{2}$$

where $y^1 \equiv t$, $y^5 \equiv s$, g^{ij} is the inverse of g_{ij} , $A^i = g^{ij}A_j$ and $N^{-2} = \phi - A^2$. The slicing lapse function is N^{-1}, while the slicing shift vector field is given by A^i.

If one is to require g_{ij} to be the metric of our four-dimensional world and A_i — the electromagnetic potentials, then N is the dilaton field and can be expressed as [4]:

$$N^2 = \frac{\det\, g}{\det\, G}\,. \tag{3}$$

The five-dimensional Kaluza metric $G_{\mu\nu}$ is a solution to the five-dimensional vacuum equations $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the five-dimensional Ricci tensor. These equations were written in terms of the extrinsic curvature $\pi_{ij} = -(N/2)(\nabla_i A_j + \nabla_j A_i - \partial_s g_{ij})$ of the four-dimensional world as follows [4, 5, 13]:

$$\begin{aligned} r_{ij} - \frac{1}{2}g_{ij}r &= N\nabla_i\nabla_j\frac{1}{N} - N(\mathcal{L}_{\mathrm{A}}\pi_{ij} + \partial_s\pi_{ij}) + (\pi\pi_{ij} - 2\pi_{ik}\pi^k_j) \\ &\qquad + \frac{1}{2}g_{ij}(\pi^2 - \pi_{kl}\pi^{kl})\,, \qquad (4) \\ 0 &= \nabla_i(\pi^i_j - \delta^i_j\pi)\,, \qquad (5) \\ \Box\frac{1}{N} &= A^i\nabla_i\pi - \frac{1}{N}\pi_{ij}\pi^{ij} - \partial_s\pi\,, \qquad (6) \end{aligned}$$

where r_{ij} is the four-dimensional Ricci tensor, r is the four-dimensional scalar curvature, ∇_i is the four-dimensional covariant derivative, $\Box = g^{ij}\nabla_i\nabla_j$, $\mathcal{L}_{\mathrm{A}}$ is the Lie derivative with respect to A_i and and $\pi = \pi^k_k$.

These equations can be equivalently written as [4, 5]:

$$r_{ij} - \frac{1}{2}g_{ij}r = \frac{N^2}{2}T_{ij}\,, \tag{7}$$

$$\nabla_i F^{ij} = -2A_i r^{ij} + \frac{2}{N^2}(\pi^{ij} - \pi^k_k g^{ij})\partial_i N\,, \tag{8}$$

$$\nabla_i(A_j\pi^{ij} - \nabla^i\frac{1}{N}) = 0\,, \tag{9}$$

$$\tag{10}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ is the Maxwell electromagnetic tensor.

The first of these equations, (7), are the equations of general relativity with matter, the second, (8), are a generalization of Maxwell's equations and the last, (9), is the gauge-fixing condition. The dilaton N is related to Newton's constant G_N via [4]:

$$\frac{N^2}{2} = \kappa = \frac{8\pi G_N}{c^4}\,. \tag{11}$$

The energy-momentum tensor T_{ij} appearing in equation (7) is given by [4, 5]:

$$T_{ij} = T^{\mathrm{Maxwell}}_{ij} + \nabla^k\Psi_{ijk} + \nabla^k\Theta_{ijk} + C_{ij} + D_{ij}\,, \tag{12}$$

where:

$$T_{ij}^{\mathrm{Maxwell}} = F_{ik}F_j{}^k - \frac{1}{4}g_{ij}F_{kl}F^{kl}\,, \tag{13}$$

$$\Psi_{ijk} = A_k\nabla_j A_i - A_j\nabla_k A_i + A_i F_{jk}\,, \tag{14}$$

$$\Theta_{ijk} = \nabla_i(A_k A_j) + g_{ij}(A^l\nabla_k A_l - A_k\nabla_l A^l)\,, \tag{15}$$

$$C_{ij} = g_{ij}A^k A^l r_{kl} - 2A^k A_j r_{ik} - 2A^k A_i r_{jk}\,, \tag{16}$$

$$\begin{aligned} D_{ij} = {} & \frac{2}{N}\nabla_i\nabla_j\frac{1}{N} - \frac{2}{N^2}\pi^k_k(A_i\partial_j N + A_j\partial_i N) \\ & + \frac{2}{N^2}\left[-A^k\pi_{ij} + A_i\pi^k_j + A_j\pi^k_i - g_{ij}(A^l\pi^k_l - A^k\pi^l_l)\right]\partial_k N. \end{aligned} \tag{17}$$

The field equations reveal a very interesting relation between the type of solution of the four-dimensional general relativity and the dilaton.

We first suppose that the dilaton is constant: $N =$ const. Let us write the five-dimensional vacuum metric $G_{\mu\nu}$ in a block-diagonal form: $G_{\mu\nu} =$ diag $(g'_{ij},\, N^{-2})$. Having $A_i = 0$ in the field equations, together with $N =$ const, results in vanishing of the full energy-momentum tensor and, therefore in a Ricci-flat four-dimensional relativity ($r_{ij} = 0$). One can re-introduce the electromagnetic potentials via a five-dimensional coordinate transformation. The only transformation which leaves the five-dimensional interval (2) invariant is: $y^i \to y^i + sc^i$, $s \to s$, where c^i are constants. Then the physical electromagnetic potentials will be $A_j = g_{jk}c^k$. Under this transformation the fields in the five-dimensional interval (2) transform as $g'_{ij} = g_{ij}$, $A'^i = A^i + c^i$, $N' = N$. Thus $r_{ij} = 0$ remains unchanged. Therefore constant dilaton and a vacuum five-dimensional Kaluza universe necessarily result in a Ricci-flat four-dimensional slice.

3 Flat Four-dimensional Universe

It is interesting to consider whether the converse is true, namely, if a Ricci-flat four-dimensional slice ($r_{ij} = 0$), embedded in a vacuum five-dimensional Kaluza universe ($R_{\mu\nu} = 0$), results in a constant dilaton ($N =$ const). We will give an example which shows that this is not the case. Consider a flat four-dimensional slice with $g_{ij} = \eta_{ij} = \mathrm{diag}(-1,\, 1,\, 1,\, 1)$. This is clearly a vacuum solution. Let us now see if the five-dimensional metric $G_{\mu\nu} = \mathrm{diag}(-1,\, 1,\, 1,\, 1,\, N^{-2})$ admits a non-constant solution for N. From the field equations we see that when $r_{ij} = 0$ and $A_i = 0$, then N must be a solution to:

$$\nabla_i\nabla_j\frac{1}{N} = 0\,. \tag{18}$$

For the flat case, the obvious solution is: $N = (a_k y^k + a_5)^{-1}$, where a_μ are constants. It will be very interesting to take:

$$a_2 = a_3 = a_4 = a_5 = 0\,, \;\; a^1 = \frac{c^2}{4\sqrt{\pi}t_0}\,. \tag{19}$$

Then Newton's constant will become $G_N = c^4 N^2/(16\pi) = (t_0/t)^2$ — the gravitational attraction falling off with time from infinity. This can be explained as a purely geometric effect between a vacuum universe embedded into another vacuum universe.
The Kasner metric [14] is:

$$d\sigma^2 = -dt^2 + \sum_{i=2}^{4} \left(\frac{t}{t_0}\right)^{2p_i} (dy^i)^2 + \left(\frac{t}{t_0}\right)^{2p_5} ds^2\,, \tag{20}$$

where $\sum_{i=2}^5\, p_i = \sum_{i=2}^5\, p_i^2 = 1$.
Solution (19) corresponds to the special case: $p_2 = p_3 = p_4 = 0\,,\, p_5 = 1$.

4 Ghost Energy-Momentum Tensor

As we are interested in solutions to the vacuum five-dimensional relativity with constant four-dimensional Newton's constant (dilaton), we will have to consider Ricci-flat solutions to four-dimensional relativity only. For $r_{ij} = 0$ and $N = \text{const}$ the field equations reduce to:

$$r_{ij} - \frac{1}{2}h_{ij}r \;=\; \frac{N^2}{2}(T_{ij}^{\text{Maxwell}} + \nabla^k\Psi_{ijk} + \nabla^k\Theta_{ijk}) \;=\; 0\,, \tag{21}$$

$$\nabla_i F^{ij} \;=\; 0\,, \tag{22}$$

$$\nabla_i(A_j\pi^{ij}) \;=\; 0\,. \tag{23}$$

We further have:

$$\nabla^j T_{ij}^{\text{Maxwell}} = F_{ik}\nabla_j F^{jk} = 0\,, \tag{24}$$

due to (22). Equation (24) is the conservation law for the energy and momentum resulting from Maxwell's equations (22).
The tensor $\nabla^k\Theta_{ijk}$ satisfies:

$$\nabla^j\nabla^k\Theta_{ijk} = -\frac{2}{N}\nabla_j\nabla_i(A_k\pi^{ik}) = 0\,, \tag{25}$$

in view of the gauge-fixing condition (23).
Considering the remaining term, $\nabla^k\Psi_{ijk}$, we see that it satisfies:

$$\nabla^j\nabla^k\Psi_{ijk} \;=\; \frac{1}{2}(\nabla^j\nabla^k + \nabla^k\nabla^j)\Psi_{ijk} + \frac{1}{2}[\nabla^j, \nabla^k]\Psi_{ijk} = 0 \tag{26}$$

in view of the antisymmetry $\Psi_{ijk} = -\Psi_{ikj}$ and $r_{ij} = 0$. Thus the tensor $\nabla^k \Psi_{ijk}$ does not describe any dynamics.
The gauge-fixing condition (23) and equation (26) lead to the conservation law:

$$\nabla^j T_{ij}^{\mathrm{Ghost}} = 0\,, \tag{27}$$

where T_{ij}^{Ghost} is the Belinfante symmetric energy-momentum tensor of the ghost fields:

$$T_{ij}^{\mathrm{Ghost}} = \nabla^k (\Psi_{ijk} + \Theta_{ijk})\,. \tag{28}$$

For the "haunted" Kaluza's universe, the energy and momentum of the ghost fields compensates completely the energy and momentum of the Maxwell's fields:

$$T_{ij}^{\mathrm{Ghost}} + T_{ij}^{\mathrm{Maxwell}} = 0 \tag{29}$$

and, therefore, it is possible to have matter co-existing with ghost matter in a Ricci flat universe.

5 Four-dimensional Lorentzian slice with Kerr Geometry

We will generate the five-dimensional solution simply by starting off with a four-dimensional static Ricci-flat solution, promoting it trivially to five dimensions (by adding ds^2 in the metric) and performing a five-dimensional coordinate transformation:

$$t \to t + \beta s\,, \tag{30}$$

where β is the inverse of the "speed of light" along the fifth, transverse dimension. This coordinate transformation will not introduce s-dependence in the four-dimensional world (as the four-dimensional metric is static and time appears only with its differential) and as a result the new *five-dimensional* "observer" will "see" the electromagnetic potentials:

$$A_j = \beta g_{tj}\,, \tag{31}$$

where $j \in \{t,\, r,\, \theta,\, \phi\}$.
The four-dimensional Kerr metric [10] in Boyer–Lindquist coordinates [15], trivially promoted to five dimensions is:

$$\begin{aligned} d\sigma^2 \;&=\; -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\; d\phi)^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)\,d\phi - a\,dt]^2 \\ &\quad + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2 + ds^2\,, \end{aligned} \tag{32}$$

where $\Delta = r^2 - \alpha r + a^2$ and $\rho^2 = r^2 + a^2 \cos^2\theta$. Here α and a are integration constants which will be identified further (in the four-dimensional Kerr solution these are the mass and the angular momentum per unit mass of a black hole). We consider the physically interesting case $\alpha > a$ (black hole solution).

The electromagnetic potentials (31) are:

$$\begin{aligned} A_t &= \beta(-1 + \frac{\alpha r}{\rho^2}), && (33)\\ A_r &= 0, && (34)\\ A_\theta &= 0, && (35)\\ A_\phi &= -\frac{a\alpha\beta r \sin^2\theta}{\rho^2}. && (36) \end{aligned}$$

For large r, the non-zero components of the vector potential are:

$$\begin{aligned} A_t &\sim \beta(-1 + \frac{\alpha}{r}), && (37)\\ A_\phi &\sim -\frac{a\alpha\beta \sin^2\theta}{r}. && (38) \end{aligned}$$

Therefore, from (37), one can identify the constant $\alpha\beta$ as electric charge:

$$q = \alpha\beta. \qquad (39)$$

Equation (38) describes the field of a magnetic dipole of strength $a\alpha\beta$, located at the origin [8, 9]:

$$m = a\alpha\beta = aq. \qquad (40)$$

Thus one can interpret the Kerr solution as a black hole generated by an electric charge and magnetic dipole (and not by a rotating massive centre). The potentials (31) satisfy the vacuum Maxwell's equations (22). The electric charge and the magnetic dipole are located at the singularity $\rho = 0$. This is the only singularity of the Kerr spacetime and can be better understood in Cartesian coordinates [16]:

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\, \sin\theta\, \cos\phi, && (41)\\ y &= \sqrt{r^2 + a^2}\, \sin\theta\, \sin\phi, && (42)\\ z &= r\cos\theta. && (43)\\ & && (44) \end{aligned}$$

The singularity $\rho = 0$, i.e. $r = 0$ and $\cos\theta = 0$, corresponds to the ring $x^2 + y^2 = a^2$.

One can analytically continue the Kerr solution for negative values of r [16]. The horizons are at:

$$r_\pm = \alpha \pm \sqrt{\alpha^2 - a^2}. \qquad (45)$$

The equations of the corresponding static horizons are:

$$r_{\pm}(\theta) = \alpha \pm \sqrt{\alpha^2 - a^2 \cos^2\theta}\,. \tag{46}$$

There are no timelike coordinates inside the ergosphere — the region between the event horizon and surrounding static horizon.
Then Kerr solution describes two universes which behave asymptotically as Schwarzschild universes — one with $r > 0$ and having a positive centre α, event horizon at r_+, and a Cauchy horizon at r_- ; the other — with $r < 0$ and having a negative centre α, event horizon at r_-, and a Cauchy horizon at r_+. In our context this has the natural interpretation of a black hole solution with positive/negative charge $q = \alpha\beta$ and a magnetic dipole of strength $m = a\alpha\beta$.

6 Four-dimensional Lorentzian slice with Taub–NUT Geometry

We consider five-dimensional Kaluza's universe with a four-dimensional Lorentzian Taub-NUT [11, 12] slice:

$$\begin{aligned} d\sigma^2 &= -V(r)(dt + 2\ell\cos\theta\, d\phi)^2 + \frac{1}{V(r)}\, dr^2 \\ &\quad + (r^2+\ell^2)(d\theta^2 + \sin^2\theta\, d\phi^2) + ds^2\,, \end{aligned} \tag{47}$$

where

$$V(r) = 1 - 2\,\frac{\alpha r + \ell^2}{r^2+\ell^2}\,, \tag{48}$$

where α and ℓ are, again, integration constants.
This metric has conical singularities at $\theta = 0,\, \pi$ (Misner string [17]). The event horizon is where $V(r)$ vanishes, i.e. at:

$$r_{\pm} = \alpha \pm \sqrt{\alpha^2 + \ell^2}\,. \tag{49}$$

The metric can be analytically continued for negative r in a similar way and we will be interested in the two regions: Region I with $\alpha \le 0$ and $r < r_- < 0$ and Region II with $\alpha \ge 0$ and $r > r_+ > 0$.
There is also an "ergoregion", surrounding the Misner string, inside of which ϕ is another timelike coordinate. The equation of these horizons is:

$$\tan^2\theta = \frac{4\ell^2 V(r)}{r^2+\ell^2}\,,\quad r < r_- \ \text{ or } \ r > r_+\,. \tag{50}$$

For the electromagnetic potentials (31) introduced with the transformation (30), we get:

$$A_t = -\beta V(r)\,, \tag{51}$$

$$A_r = 0\,, \tag{52}$$
$$A_\theta = 0\,, \tag{53}$$
$$A_\phi = -2\ell\beta V(r)\cos\theta\,. \tag{54}$$

Asymptotically, for large r we have:

$$A_t = -\beta\left(1-\frac{2\alpha}{r}\right)\,, \tag{55}$$
$$A_\phi = -2\ell\beta\left(1-\frac{2\alpha}{r}\right)\cos\theta\,. \tag{56}$$

Equation (55) is the electric potential due to charge

$$q = 2\alpha\beta\,. \tag{57}$$

Equation (56) is the potential due to a magnetic monopole of charge $m = 2\beta\ell$. This can be seen by integrating the flux of the magnetic field B^r through the infinite sphere:

$$4\pi m = \lim_{r\to\infty}\oint_{S_r} B^r\, ds_r = \lim_{r\to\infty}\oint_{S_r} F_{\theta\phi}\, d\theta d\phi = 4\pi(2\beta\ell)\,. \tag{58}$$

For Kerr geometry this integral vanishes (we do not have a monopole but a magnetic dipole there).
The location of these charges appears to be unclear as $V(r) = 1 - 2(\alpha r + \ell^2)(r^2+\ell^2)^{-1}$, appearing in (51) and (54), is not singular outside the horizons. However, every spacelike hypersurface which is pushed between the horizons becomes singular [18] and therefore, we can interpret the points $ir = \pm\ell$ (note that ir is a spacelike coordinate between the horizons) as the loci where the images of the charges are seen by an observer with $r > r_+$ or $r < r_-$. For the case $\alpha = 0$, the proper distance between the origin and the location of the images is:

$$\int_{i0}^{\pm i\ell} \frac{dr}{\sqrt{V(r)}} = \pm\,\ell \int_0^{\ln(1+\sqrt{2})} \sqrt{1-\sinh^2 x}\, dx \;\approx\; \pm 0.7\ell\,. \tag{59}$$

Therefore, an observer with $r > r_+ > 0$ (Region I) will register the image of a monopole a proper distance $|0.7\ell|$ from the origin, while an observer with $r < r_- < 0$ (Region II) will register the image of a monopole a proper distance $-|0.7\ell|$ from the origin.

7 Conclusions

We have presented solutions to the original Kaluza's theory which describe static electric and magnetic fields generated by point-like electric and magnetic charges and dipoles. Unlike the dual Kaluza–Klein theory (namely,

Klein's modification of the original Kaluza's theory) these solutions allow to have a constant Newton's constant, as the gauge degrees of freedom are now transferred to the dilaton. The gauge-fixing of the electromagnetic potentials results in the appearance of a Belinfante ghost part in the full energy-momentum tensor which fully compensates the electromagnetic energy-momentum tensor. Four-dimensional solutions with vanishing Ricci tensor ($r_{ij} = 0$) are the only possible solutions when the dilaton is required to be constant in a Ricci-flat five-dimensional universe. These four-dimensional gravitational Ricci-flat solutions can be interpreted from a five-dimensional Kaluza's perspective, as solutions generated by four-dimensional electromagnetism of charges and dipoles or their images (for the case of a Taub–NUT four-dimensional slice). The integration constants $a\,,\alpha$ and β (for Kerr geometry) and $\alpha\,,\ell$ and β (for Taub–NUT geometry) have interpretation as charges (see (39) and (40) for Kerr geometry and (57) and (58) for Taub–NUT geometry) and the solutions represent gravitational attraction without unphysical regions with gravitational repulsion, as, for example, in the Reissner–Nordstrøm case [19].

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